On the deductive closure of a partially ordered propositional belief base

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Résumé

Dans ce papier, nous présentons des résultats pour l'extension de la logique possibiliste dans un cadre partiellement ordonné. Le point difficile réside dans le fait que les définitions dans le cas totalement ordonné ne sont plus équivalentes dans le cas partiellement ordonné. Nous commençons par rappeler la logique possibiliste avec poids symboliques. Par contraste, nous considérons des bases propositionnelles munies d'une relation d'ordre partiel, exprimant la certitude relative. Une sémantique possible consiste à supposer que cet ordre provient d'un ordre partiel sur les modèles. Elle exige la capacité d'induire un ordre partiel sur les sous-ensembles d'un ensemble, à partir d'un ordre partiel sur ses éléments. Parmi plusieurs définitions de relations d'ordre partiel ainsi définies, nous sélectionnons la plus intéressante. Nous montrons les limites d'une sémantique basée sur un ordre partiel unique sur les modèles et proposons une sémantique plus générale. Nous utilisons un langage de plus haut niveau qui exprime des conjonctions de paires de formules en relation, avec des axiomes qui décrivent les propriétés de la relation. Nous proposons également deux approches syntaxiques pour inférer de nouvelles paires de formules et compléter l'ordre sur le langage propositionnel et nous les comparons à la logique possibiliste avec poids symboliques.

Abstract

This paper presents results toward the extension of possibilistic logic in a partially ordered case. The difficult point lies in the fact that equivalent definitions in the totally ordered case are no longer equivalent in the partially ordered one. We start by recalling possibilistic logic with symbolic weights. In contrast we use logical bases equipped with a partial order expressing relative certainty. A possible semantics consists in assuming the partial order on formulas stems from a partial order on interpretations. It requires the capability of inducing a partial order on subsets of a set from a partial order on its elements. Among different possible definitions of partial order relations, we select the most interesting one. We show the limitations of a semantics based on a unique partial order on interpretations and propose a more general semantics. At the syntactic level we use a language expressing pairs of related formulas and axioms describing the properties of the ordering. We propose two syntactic inference methods in order to get a partial order on the whole propositional language. We compare these methods with inference in possibilistic logic with symbolic weights.

1 Introduction

Possibilistic logic [10] is an approach to reasoning under uncertainty based on ranked propositional bases. The rank of a formula, often encoded by a weight in $(0, 1]$, is understood as its degree of certainty. Degrees of uncertainty follow the rules of accepted belief, namely that believing each of two formulas to the same degree is equivalent to believing their conjunction to that degree. The deductive closure of a possibilistic logic base comes down to a ranking of the classical closure of the base without the weights. It follows the rule of the weakest link : the strength of a chain of inference steps is the one of the least certain formula involved in the chain. The weight of a formula in the deductive closure is the weight of the most reliable inference path from the belief base to this formula. Possibilistic logic has proved instrumental in the representation and reasoning techniques for various domains including non-monotonic reasoning, belief revision and fusion [9].

Ranked knowledge bases naturally appear when processing sets of default rules according to their specificity, or when information comes from several more or less reliable sources.

In the last 10 years, the question whether these results can be extended to partially ordered bases has been debated, and various approaches have been proposed [1, 16, 2]. Among these approaches, the one based on partially ordered symbolic weights [2] appears as very natural and amenable to proof methods. Independently, starting with Lewis [14], some conditional logics have been proposed to reason with pairs of formulas related by a connective expressing relative certainty or possibility, in the framework of total possibility orderings. Besides, Halpern [12] has suggested such a logic in the case of partial orders between formulas obtained from partial orders on interpretations.

Following the line opened by Halpern, this paper tries to give a language, a semantics and a proof method for reasoning with partially ordered belief bases, and to compare this type of framework with the one of possibilistic logic with symbolic weights where the partial order is bearing on weights.

2 Background on possibilistic logic

In this section we recall the construction of possibilistic logic, useful in the sequel.

2.1 Possibilistic base

Let $\mathcal{L}$ be a propositional language, where the formulas are denoted by $\phi_1 \cdots \phi_n$, and Ω the set of interpretations. $[\phi]$ denotes the set of models of ϕ, a subset of Ω.
Possibilistic logic is an extension of classical logic which handles weighted formulas of the form (ϕ_j, a_j) where ϕ_j is a propositional formula and $a_j \in]0, 1]$. (ϕ_j, a_j) is interpreted by $N([\phi_j]) \geq a_j$, where N is a necessity measure [10]. A possibility measure is defined on subsets of Ω from a possibility distribution π on Ω as $\Pi(A) = \max_{\omega \in A} \pi(\omega)$ expressing the plausibility of any proposition ϕ, with $[\phi] = A$, and the necessity measure expressing certainty levels is defined by $N(A) = 1 - \Pi(A^c)$ where A^c is the complement of A. So a_j can be seen as a degree of certainty.

A possibilistic base is a finite set of weighted formulas $\Sigma = \{(\phi_j, a_j), j = 1 \cdots m\}$. It can be associated with a possibility distribution π_Σ on Ω in the following way :

- $\forall j, \pi_j(\omega) = \begin{cases} 1 \text{ if } \omega \in [\phi_j], \\ 1 - a_j \text{ if } \omega \notin [\phi_j]. \end{cases}$
- $\pi_\Sigma(\omega) = \min_j \pi_j(\omega)$.

Note that π_j is the least informative possibility distribution among those such that $N([\phi_j]) \geq a_j$, where a possibility distribution π is less informative than ρ if and only if $\pi \geq \rho$. Likewise π_Σ is the least informative possibility distribution compatible with the base Σ. This is a basic component of possibility theory called *the principle of minimal specificity*. It can be indeed checked that $N_\Sigma(\phi_j) = \min_{\omega \notin [\phi]}(1 - \pi_\Sigma(\omega)) \geq a_j$, and this is the least necessity degree in agreement with Σ. However, it may occur that $N_\Sigma(\phi_j) > a_i$. The (semantic) closure of Σ is then defined by $\{(\phi, b) : \phi \in \mathcal{L}, N_\Sigma(\phi) = b > 0\}$, which comes down to a ranking on the language. A possibilistic base $\Sigma = \{(\phi_j, a_j), j = 1 \ldots m\}$ such that $a_j = N_\Sigma(\phi_j), \forall j = 1, \ldots m$ is called *coherent* (by analogy with coherent lower probabilities [15]).

2.2 Syntactic inference

A sound and complete syntactic inference can be defined with the following axioms and inference rules [8] :

Axioms of classical logic are turned into formulas weighted by 1 :

- $(\phi \to (\psi \to \phi), 1)$
- $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)), 1)$
- $((\neg\phi) \to (\neg\psi)) \to (\psi \to \phi), 1)$

The inference rules are :

- Weakening rule : If $a > b$ then $(\phi, a) \vdash (\phi, b)$
- Modus Ponens : $\{(\phi \to \psi, a), (\phi, a)\} \vdash (\psi, a)$

This Modus Ponens rule embodies the law of accepted beliefs at any level, assumed they form a deductively closed set. It is related to axiom K of modal logic.

A possibilistic base can be inconsistent to a degree less that 1. The above inference system enables to define the degree of inconsistency of a possibilistic base as the highest certainty degree to which an inconsistency can be obtained :

$$Inc(\Sigma) = \max\{a | \Sigma \vdash (\bot, a)\}.$$

It can be proved that $Inc(\Sigma) = 1 - \max_{\omega \in \Omega} \pi_\Sigma(\omega)$, and that $N_\Sigma(\phi) = Inc(\Sigma \cup (\neg\phi, 1))$.

2.3 Ordinal semantics of a possibilistic base

In place of a possibility distribution, we can define the semantics of the possibilistic consequence as a total preorder on Ω.

Let ω be an interpretation. $\overline{\Sigma(\omega)}$ will denote the set of formulas of Σ falsified by ω. Let $\geq_\Sigma$ be the preorder on Ω defined by :

$$\omega \geq_\Sigma \omega' \text{ if and only if } \\ \forall(\phi_j, a_j) \in \Sigma \text{ such that } \phi_j \in \overline{\Sigma(\omega)}, \\ \exists(\phi_i, a_i) \in \Sigma \text{ such that } \phi_i \in \overline{\Sigma(\omega')} \text{ and } a_i \geq a_j.$$

Then it is easy to see that $\pi_\Sigma(\omega) \geq \pi_\Sigma(\omega')$ if and only if $\omega \geq_\Sigma \omega'$, so that $\geq_\Sigma$ is a possibility ordering.

The above preorder enables to build an ordered deductive closure as follows : $\phi \succ_N \psi$ iff $\forall\omega \in \overline{[\phi]}$, $\exists\omega' \in \overline{[\psi]}$ and $\omega' >_\Sigma \omega$. It can be proved that $\phi \succ_N \psi$ iff $N_\Sigma(\phi) > N_\Sigma(\psi)$, which means that we recover the ranking on the whole language.

3 Possibilistic logic with symbolic weights

Possibilistic logic has been extended to the case when only partial information is available about the ordering between the otherwise unknown weights of the formulas [2].

3.1 Symbolic possibilistic base

Propositional formulas are now associated with symbolic weights, and a set of constraints on these weights expresses the relative strength of these weights. Given a set of symbols $H = \{a, b, \ldots\}$ interpreted as variables on the scale $]0, 1]$, composite symbolic weights w are construed as combinations of such variables using $\max, \min$.

Let $\Sigma = \{(\phi_j, w_j), j = 1 \cdots m\}$ be a symbolic possibilistic base where w_j is a composite symbolic weight built on H. A formula (ϕ_j, w_j) is still interpreted as $N(\phi_j) \geq w_j$.
Benferhat et al. [2] consider a set of constraints C of the form $w_i \geq w_j$, or $w_i > w_j$ where w_i and w_j are composite symbolic weights. They define $C \vDash w > w'$ iff every numerical valuation (in $]0, 1]$) that satisfies the constraints of C also satisfies $w \geq w'$, and at least one satisfies $w > w'$. This definition of $w > w'$ looks too liberal (for instance, it would imply $\max(a, b) > b$).

In this paper, we restrict to symbolic possibilistic bases with elementary weights $a, b, \ldots$, and C contains strict dominance statements only between such weights forming a partial order. We define $C \vDash w > w'$ (resp. $C \vDash w \geq w'$) iff every numerical valuation (in $]0, 1]$) that satisfies the constraints of C also satisfies $w > w'$ (resp. $w \geq w'$).

In fact, the use of the principle of minimal specificity will tend to turn constraints $w \geq w'$ into $w = w'$ whenever neither $w > w'$ nor $w' > w$ can be proven.

3.2 Inference in symbolic possibilistic base

As in possibilistic logic, in order to get the deductive closure, we must compute $N_\Sigma(\phi), \forall \phi \in \mathcal{L}$. However, to do it, one must slightly modify the axiom system of possibilistic logic and allow the symbolic handling of max and min. So the two above inference rules of standard possibilistic logic are replaced by

- Fusion rule : $\{(\phi, w), (\phi, w')\} \vdash (\phi, \max(w, w'))$
- Weighted Modus Ponens : $\{(\phi \to \psi, w), (\phi, w')\} \vdash (\psi, \min(w, w'))$

involving a symbolic handling of weights. If Σ^* is the set of formulas appearing in Σ, and B is a subset of Σ^* that implies ϕ, it is clear that $(\Sigma, C) \vdash (\phi, \min_{\phi_j \in B} w_j)$, and so

$$N_\Sigma(\phi) = \max_{B \subseteq \Sigma^*, B \vdash \phi} \min_{\phi_j \in B} w_j.$$

Note that in the above expression, we can restrict the maximization to minimal subsets B implying ϕ.

Checking $w > w'$ comes down to comparing such kinds of max-min expressions, using knowledge in C. We still have that $N_\Sigma(\phi) = Inc(\Sigma \cup (\neg\phi, 1))$.

Example 1
$\Sigma = \{(\phi, a), (\neg\phi \vee \psi, b), (\neg\psi, c)\}$, *with* $C = \{a > c, b > c\}$. *We can infer* ψ *from* Σ *:* $(\Sigma, C) \vdash (\psi, \min(a, b))$.

We can notice that the above consequence just allows us to deduce a formula with a symbolic composite weight. It could be extended to produce pairs of formulas of the form $\phi > \psi$. The idea is to deduce ϕ (resp. ψ) with its symbolic weight w (resp. w') and to compare these weights.
Formally, let (Σ, C) a symbolic possibilistic base, and ϕ and ψ two formulas, we define :

Definition 1 $(\Sigma, C) \vdash \phi > \psi$ *iff* $N_\Sigma(\phi) > N_\Sigma(\psi)$

4 Partially ordered belief bases and partial orders on models

Let $(\mathcal{K}, >)$ be a partially ordered finite set of formulas of $\mathcal{L}$. We associate with $(\mathcal{K}, >)$ a list of statements of the form $\phi > \psi$, where no weight appears explicitly, and $>$ is asymmetric. In possibilistic logic, two interpretations are compared by considering their falsified formulas, and two formulas are then compared by considering their countermodels. This ordinal construction outlined in Section 2.3 can be generalized to arbitrary partially ordered bases for building a deductive closure. However, it requires the capability of inducing a partial order on subsets of a set from a partial order on its elements. Properties of partial orders among sets have been studied in [4] and arguments have been given for selecting one approach extending comparative possibility, the weak optimistic dominance.

In this section, we first recall the properties of the weak optimistic dominance. Then, we study the properties of the deductive closure which can be built from it and show its limitations.

4.1 Weak optimistic dominance

Let $(S, >)$ be a partially ordered set, where $>$ is asymmetric and transitive. There are various possible definitions for building a relation that compares subsets A and B of S. These relations have been studied by Halpern [12], and more extensively by the authors [4]. Here we are interested to the generalisation of possibility and necessity measures to strict partial possibility orderings. The following extensions are thus of interest :

1. Weak optimistic strict dominance :
   $A \succ_{wos} B$ iff $A \neq \emptyset$ and $\forall b \in B,\ \exists a \in A, a > b$.
2. Strong optimistic strict dominance : $A \succ_{Sos} B$ iff $\exists a \in A,\ \forall b \in B,\ a > b$

It is clear that if $(S, >)$ is the strict part of a complete preordering encoded by a possibility distribution π, $A \succ_{wos} B$ if and only if $A \succ_{Sos} B$ if and only if $\Pi(A) > \Pi(B)$, which we can denote $>_\Pi$. This is the comparative possibility of Lewis [14].

The following properties are obviously valid for the relation $>_\Pi$ and its weak form $\geq_\Pi$ between subsets :

- **Compatibility with Inclusion (CI)** If $B \subseteq A$ then $A \geq_\Pi B$
- **Orderliness (O)** If $A >_\Pi B$, $A \subseteq A'$, and $B' \subseteq B$, then $A' >_\Pi B'$
- **Stability for Union (SU)** If $A \geq_\Pi B$ then $A \cup C \geq_\Pi B \cup C$
- **Qualitativeness (Q)** If $A \cup B >_\Pi C$ and $A \cup C >_\Pi B$, then $A >_\Pi B \cup C$
- **Negligibility (N)** If $A >_\Pi B$ and $A >_\Pi C$, then $A >_\Pi B \cup C$
- **Conditional Closure by Implication (CCI)** If $A \subseteq B$ and $A \cap C >_\Pi \overline{A} \cap C$ then $B \cap C >_\Pi \overline{B} \cap C$
- **Conditional Closure by Conjunction (CCC)** If $C \cap A >_\Pi C \cap \overline{A}$ and $C \cap B >_\Pi C \cap \overline{B}$ then $C \cap (A \cap B) >_\Pi C \cap \overline{A \cap B}$
- **Left Disjunction (OR)** If $A \cap C >_\Pi A \cap \overline{C}$ and $B \cap C >_\Pi B \cap \overline{C}$ then $(A \cup B) \cap C >_\Pi (A \cup B) \cap \overline{C}$
- **Cut (CUT)** If $A \cap B >_\Pi A \cap \overline{B}$ and $A \cap B \cap C >_\Pi A \cap B \cap \overline{C}$ then $A \cap C >_\Pi A \cap \overline{C}$
- **Cautious Monotony (CM)** : If $A \cap B >_\Pi A \cap \overline{B}$ and $A \cap C >_\Pi A \cap \overline{C}$ then $A \cap B \cap C >_\Pi A \cap B \cap \overline{C}$

In fact, stability for union (along with obvious non triviality properties such as $S >_\Pi \emptyset$ and $A \geq_\Pi \emptyset$), are enough to characterize these relations [6].

The following properties have been established for the strict orderings $\succ_{wos}$ and $\succ_{Sos}$ [12, 4] :

Proposition 1 *The weak optimistic strict dominance $\succ_{wos}$ is a strict partial order which satisfies Qualitativeness (Q), Orderliness (O), Negligibility (N), Conditional Closure by Implication (CCI), Conditional Closure by Conjunction (CCC), Left Disjunction (OR), (CUT), (CM), and the converse of (SU) in the form : if $A \cup C \succ_{wos} B \cup C$ then $A \succ_{wos} B$.*

The strong optimistic strict dominance $\succ_{Sos}$ is a strict order satisfying Orderliness (O) and Cautious Monotony (CM) . However it fails to satisfy Negligibility, Qualitativeness, CUT and Left Disjunction (OR),

It is clear that the weak optimistic strict dominance $\succ_{wos}$ is the most promising extension of the comparative possibility relation to partial orders.

Another way to induce a partial order on 2^S from a partial order $>$ on S is to consider the partial order $>$ as a family of total orders $> i$ extending (or compatible with) this partial order. Let A and B two subsets of S, and let $>^i_\Pi$ denote the partial order on 2^S induced by $>^i$. Then :

Proposition 2 *Let A, B two subsets of S. We have :*

$$A \succ_{wos} B \iff \forall i = 1..n\ A >^i_\Pi B$$

As a consequence, the weak optimistic strict order on subsets is characterised by several total orderings on elements. Given the properties satisfied by $\succ_{wos}$, this result clearly bridges the gap between the weak optimistic dominance and the partially ordered non-monotonic inference setting of Kraus, Lehmann and Magidor [13] interpreting the dominance $A \succ_{wos} B$ when $A \cap B = \emptyset$ as the default inference of A from $A \cup B$.

4.2 Weak optimistic deductive closure

Let $\mathcal{K}$ be a finite set of formulas of $\mathcal{L}$. $\mathcal{K}(\omega)$ (resp. $\overline{\mathcal{K}(\omega)}$) denotes the subset of formulas of $\mathcal{K}$ satisfied (resp. falsified) by ω.

Principle : The deductive closure $\succ_N$ proposed in [4] is constructed in two steps, applying twice the extension of a partial order on the elements to subsets with $\succ_{wos}$. Like the procedure in Section 2.3 for possibilistic logic, it defines the dominance on interpretations in terms of the violation of the most certain formulas. But here, these formulas may be incomparable.

Definition 2 (Partial-order semantics)
From $(\mathcal{K}, >)$ **to**$(\Omega, \rhd)$ **:** $\forall \omega, \omega' \in \Omega$, $\omega \rhd \omega'$ *iff* $\overline{\mathcal{K}(\omega')} \succ_{wos} \overline{\mathcal{K}(\omega)}$
From $(\Omega, \rhd)$ **to** $(\mathcal{L}, \succ_N)$ $\forall \phi, \psi \in \mathcal{L}, \phi \succ_N \psi$ *iff* $\overline{[\psi]} \rhd_{wos} \overline{[\phi]}$.

In the case of a total order, this amounts to defining a relationship of necessity [10].

Definition 3 *The possibilistic deductive closure $\mathcal{C}_N(\mathcal{K}, >)$ of the partially ordered set $(\mathcal{K}, >)$ is defined as follows :*

$$\mathcal{C}_N(\mathcal{K}, >) = \{(\phi, \psi) \in \mathcal{L}^2, \phi \succ_N \psi\}.$$

We write $\mathcal{K} \models \phi \succ_N \psi$ whenever $(\phi, \psi) \in \mathcal{C}_N(\mathcal{K}, >)$.

In agreement with [7], one may extract from $\mathcal{C}_N(\mathcal{K}, >)$ the set of accepted beliefs when ϕ is known to be true as : $\mathcal{A}_\phi(\mathcal{K}, >)_{\succ_N} = \{\psi : (\phi \to \psi, \phi \to \neg\psi) \in \mathcal{C}_N(\mathcal{K}, >)\}$.

The properties of the relation $\succ_{wos}$ can be used to obtain the properties satisfied by the relation $\succ_N$. The following result is easy to prove :

Proposition 3 *Let ϕ, ψ two formulas of $\mathcal{K}$. Let $A = [\neg\phi]$ and $B = [\neg\psi]$. Then :*
$\phi \to \psi \succ_N \phi \to \neg\psi$ iff $[\phi] \cap [\psi] \rhd_{wos} [\phi] \cap \overline{[\psi]}$

Proof of Proposition 3: $\phi \to \psi \succ_N \phi \to \neg\psi$ iff $[\phi \wedge \psi] \rhd_{wos} [\phi \wedge \neg\psi]$ iff $[\phi] \cap [\psi] \rhd_{wos} [\phi] \cap \overline{[\psi]}$. □

Let P a property of the relation $\succ_{wos}$. The conjugated property P' is a property of the relation $\succ_N$ such that $\succ_{wos}$ satisfies P iff $\succ_N$ satisfies P'.

Proposition 4 *The relation $\succ_N$ satisfies the following properties :*

Q' *: If $\chi > \phi \wedge \psi$ and $\psi > \phi \wedge \chi$ then $\psi \wedge \chi > \phi$*

O *: If $\phi > \psi$, $\phi \vDash \phi'$ and $\psi' \vDash \psi$ then $\phi' > \psi'$*

ADJ *: If $\psi > \phi$ and $\chi > \phi$ then $\psi \wedge \chi > \phi$*

OR' *: If $\phi \rightarrow \chi > \phi \rightarrow \neg\chi$ and $\psi \rightarrow \chi > \psi \rightarrow \neg\chi$ then $(\phi \vee \psi) \rightarrow \chi > (\phi \vee \psi) \rightarrow \neg\chi$*

CCC' *: If $\chi \rightarrow \phi > \chi \rightarrow \neg\phi$ and $\chi \rightarrow \psi > \chi \rightarrow \neg\psi$ then $\chi \rightarrow (\phi \wedge \psi) > \chi \rightarrow \neg(\phi \wedge \psi)$.*

CUT' *: If $\phi \rightarrow \psi > \phi \rightarrow \neg\psi$ and $(\phi \wedge \psi) \rightarrow \chi > (\phi \wedge \psi) \rightarrow \neg\chi$ then $\phi \rightarrow \chi > \phi \rightarrow \neg\chi$*

CM' *: If $\phi \rightarrow \psi > \phi \rightarrow \neg\psi$ and $\phi \rightarrow \chi > \phi \rightarrow \neg\chi$ then $(\phi \wedge \psi) \rightarrow \chi > (\phi \wedge \psi) \rightarrow \neg\chi$.*

Example 2 *Let $(\mathcal{K}, >) = \{\neg x \vee y > x \wedge y,\ x \wedge y > \neg x,\ x > \neg x\}$ be a partially ordered base.*

- *From $(\mathcal{K}, >)$ to $(\Omega, \rhd)$: we obtain $xy \rhd_{wos} \{\overline{x}y, x\overline{y}, \overline{xy}\}$. This partial order is compatible with 6 possible complete orders :*
  - $xy \rhd \overline{x}y \rhd x\overline{y} \rhd \overline{xy}$
  - $xy \rhd \overline{x}y \rhd \overline{xy} \rhd x\overline{y}$
  - $xy \rhd \overline{xy} \rhd \overline{x}y \rhd x\overline{y}$
  - $xy \rhd \overline{xy} \rhd x\overline{y} \rhd \overline{x}y$
  - $xy \rhd x\overline{y} \rhd \overline{xy} \rhd \overline{x}y$
  - $xy \rhd x\overline{y} \rhd \overline{x}y \rhd \overline{xy}$
- *From $(\Omega, \rhd)$ to $(\mathcal{L}, \succ_N)$: we obtain $x \succ_N \neg x$, $x \wedge y \succ_N \neg x$ and $\neg x \vee y \succ_N \neg x$ but not $\neg x \vee y \succ_N x \wedge y$*

So $(\mathcal{K}, >) \not\subseteq \mathcal{C}_N(\mathcal{K}, >)$ violating one of Tarski's axioms ($A \subseteq C(A)$).

In the final order over formulas, $\neg x \vee y$ and $x \wedge y$ are incomparable. The reason is that some information has been lost when going from $(\mathcal{K}, >)$ to $(\Omega, \rhd)$. Indeed, if the strict partial order $>$ on the base $\mathcal{K}$, is interpreted as part of a strict relation $\succ_N$ of necessity, the application of Definition 2 produces the following constraints :

- *Due to $\neg x \vee y \succ_N x \wedge y$ we must have ($\overline{x}y \rhd x\overline{y}$ or $\overline{xy} \rhd x\overline{y}$)*
- *Due to $x \wedge y \succ_N \neg x$ we must have ($xy \rhd x\overline{y}$) and ($xy \rhd \overline{x}y$ or $x\overline{y} \rhd \overline{x}y$) and ($xy \rhd \overline{xy}$ or $x\overline{y} \rhd \overline{xy}$)*
- *Due to $x \succ_N \neg x$ we must have ($xy \rhd \overline{x}y$ or $x\overline{y} \rhd \overline{x}y$) and ($xy \rhd \overline{xy}$ or $x\overline{y} \rhd \overline{xy}$)*

It is easy to see that these constraints imply this partial order : $xy \rhd \{\overline{x}y, x\overline{y}, \overline{xy}\}$ and ($\overline{x}y \rhd x\overline{y}$ or $\overline{xy} \rhd x\overline{y}$). We obtain a stronger condition that the partial order $\rhd_{wos}$, which is compatible with only 4 complete orders :

- $xy \rhd \overline{x}y \rhd x\overline{y} \rhd \overline{xy}$
- $xy \rhd \overline{xy} \rhd \overline{x}y \rhd x\overline{y}$
- $xy \rhd \overline{x}y \rhd \overline{xy} \rhd x\overline{y}$
- $xy \rhd \overline{xy} \rhd x\overline{y} \rhd \overline{x}y$

The impossibility of representing the partial order $(\mathcal{K}, >)$ by a partial order on interpretations is the cause for losing the piece of information $\neg x \vee y > x \wedge y$. It suggests that the possibilistic deductive closure $\mathcal{C}_N(\mathcal{K}, >)$ is too weak to account for semantic entailment in partially ordered bases.

5 A sound and complete approach to deduction with partially ordered bases

As shown in the previous section, the deductive closure built from assuming that a partial order on formulas can be expressed by a partial order on models does not always preserve the initial ordering of the base. We propose to use a stronger semantics, and to use an inference system based on the properties of the partial order, and the use of formulas of the form $\phi > \psi$. The idea is to define a logic for reasoning about partially ordered bases, with inference rules inspired from the properties of the relation $\succ_N$.
After presenting the language, we propose an inference system with axioms and inference rules. Then we propose a semantics in terms of a preorder over sets of interpretations for which the inference system is sound and complete. Lastly, we propose another weaker, but convenient syntactic inference method based on level cuts of the initial base and classical logic.

5.1 The inference system $\mathcal{S}$

5.1.1 Syntax

$\mathcal{L}$ denotes a classical propositional language, where formulas denoted by $\phi,\ \psi, \cdots$ are built using usual connectives $\neg, \wedge, \vee$ of classical logic and atoms. $>$ denotes a strict partial order on formulas of $\mathcal{L}$. The main idea is to encapsulate the language $\mathcal{L}$ in a language equipped with a partial order relation, denoted by $\mathcal{L}_>$.
Formally, a literal Φ of $\mathcal{L}_>$ is of the form $\phi > \psi$ or $\neg(\phi > \psi)$, ϕ and ψ being formulas of $\mathcal{L}$.
A formula of $\mathcal{L}_>$ is either a literal Φ of $\mathcal{L}_>$, or a formula of the form $\Psi \wedge \Gamma$ with $\Psi,\ \Gamma \in \mathcal{L}_>$.
We associate with $(\mathcal{K}, >)$ the conjunction of formulas of $\mathcal{L}_>$ of the form $\phi > \psi$, where $\phi, \psi \in \mathcal{K}$/

5.1.2 Axioms and inference rules

The idea is to use as axioms and inference rules the basic properties of the relation $\succ_N$.

ax_1 : $\top > \bot$.

ax_2 : If $\psi \vDash \phi$ then $\neg(\psi > \phi)$.

RI_1 : If $\chi > \phi \wedge \psi$ and $\psi > \phi \wedge \chi$ then $\psi \wedge \chi > \phi$ (Q').

RI_2 : If $\phi > \psi$, $\phi \vDash \phi'$ and $\psi' \vDash \psi$ then $\phi' > \psi'$ (O).

RI_3 : If $\phi > \psi$ and $\psi > \chi$ then $\phi > \chi$ (T).

RI_4 : If $\phi > \psi$ then $\neg(\psi > \phi)$ (NR).

The first axiom says that the order relation is not trivial [1]. The second one that the order relation does not contradict classical inference. Rules RI_1 and RI_2 correspond to the properties of Qualitativeness and Orderliness. Rules RI_3 and RI_4 express the transitivity and irreflexivity of the relation $\succ_{wos}$. We call $\mathcal{S}$ this inference system.

Other inference rules can be derived from the above rules. We have shown in [4] that the properties O et Q allow to recover other properties of the partial relation $\rhd_{wos}$. Thus using conjugate relations, the following inference rules can be produced :

RI_5 : If $\psi > \phi$ and $\chi > \phi$ then $\psi \wedge \chi > \phi$ (ADJunction).

RI_6 : If $\phi \to \chi > \phi \to \neg\chi$ and $\psi \to \chi > \psi \to \neg\chi$ then $(\phi \vee \psi) \to \chi > (\phi \vee \psi) \to \neg\chi$ (OR').

RI_7 : If $\chi \to \phi > \chi \to \neg\phi$ and $\chi \to \psi > \chi \to \neg\psi$ then $\chi \to (\phi \wedge \psi) > \chi \to \neg(\phi \wedge \psi)$ (CCC').

RI_8 : If $\phi \to \psi > \phi \to \neg\psi$ and $(\phi\wedge\psi) \to \chi > (\phi\wedge\psi) \to \neg\chi$ then $\phi \to \chi > \phi \to \neg\chi$ (CUT').

RI_9 : If $\phi \to \psi > \phi \to \neg\psi$ and $\phi \to \chi > \phi \to \neg\chi$ then $(\phi \wedge \psi) \to \chi > (\phi \wedge \psi) \to \neg\chi$ (CM').

RI_{10} : If $\phi > \bot$ then $\phi > \neg\phi$

In the following, $(\mathcal{K}, >) \vdash_{\mathcal{S}} \Phi$ denotes that Φ is a consequence of the partially ordered set $\mathcal{K}$ in the inference system $\mathcal{S}$.

5.2 Semantics

We consider a semantics defined by a relation between sets of interpretations (instead of interpretations). We formally define the semantics before proving that the inference system $\mathcal{S}$ is sound and complete for that semantics. The idea is to interpret the formula $\phi > \psi$ on 2^Ω by $\overline{[\psi]} \rhd \overline{[\phi]}$.

Definition 4
- *A model M is a structure $(2^\Omega, \rhd)$ where $\rhd$ is a strict partial order on 2^Ω satisfying the properties O and Q* [2].
- $M \models_{\mathcal{S}} (\phi > \psi)$ *iff* $\overline{[\psi]} \rhd \overline{[\phi]}$.

We extend the semantic consequence $\models_{\mathcal{S}}$ to conjunctions or negations of formula $\phi > \psi$ as usual.
Given $(\mathcal{K}, >)$ a finite partially ordered base, we can write $(\mathcal{K}, >) = \{(\phi_i > \psi_i), i = 1 \cdots n\}$. So we have $M \models_{\mathcal{S}} (\mathcal{K}, >)$ iff $\forall i = 1 \cdots n, \overline{[\psi_i]} \rhd \overline{[\phi_i]}$.
Then we define $(\mathcal{K}, >) \models_{\mathcal{S}} \phi > \psi$ by $\forall M$, if $M \models_{\mathcal{S}} (\mathcal{K}, >)$ then $M \models_{\mathcal{S}} (\phi > \psi)$, which can also be written :
For any strict order $\rhd$ on 2^Ω satisfying O and Q, if $\forall i = 1 \cdots n, \overline{[\psi_i]} \rhd \overline{[\phi_i]}$ then $\overline{[\psi]} \rhd \overline{[\phi]}$.

Proposition 5 *Let $(\mathcal{K}, >)$ a partially ordered base*
- ***Soundness :***
  $(\mathcal{K}, >) \vdash_{\mathcal{S}} \phi > \psi \Rightarrow (\mathcal{K}, >) \models_{\mathcal{S}} \phi > \psi$
- ***Completeness :***
  $(\mathcal{K}, >) \models_{\mathcal{S}} \phi > \psi \Rightarrow (\mathcal{K}, >) \vdash_{\mathcal{S}} \phi > \psi$

Proof of Proposition 5:
Let $(\mathcal{K}, >) = \{(\phi_i > \psi_i), i = 1 \cdots n\}$.
- **Soundness :**
  Let $\rhd$ be a strict partial order on 2^Ω satisfying O and Q. We must show that if $\forall i = 1 \cdots n, \overline{[\psi_i]} \rhd \overline{[\phi_i]}$ then $\overline{[\psi]} \rhd \overline{[\phi]}$. Assume that $\phi > \psi$ was obtained from the $(\phi_i > \psi_i)$ by inference rules RI_1, RI_2 and RI_3. It remains to prove that each of the rules is sound.

  RI_1 We must show that if $\overline{[\phi \wedge \psi]} \rhd \overline{[\chi]}$ and $\overline{[\phi \wedge \chi]} \rhd \overline{[\psi]}$ then $\overline{[\phi]} \rhd \overline{[\psi \wedge \chi]}$.This is true since the relation $\rhd$ satisfies Q.

  RI_2 We must show that if $\overline{[\psi]} \rhd \overline{[\phi]}$, $\phi \models \phi'$ and $\psi' \models \psi$ then $\overline{[\psi']} \rhd \overline{[\phi']}$. This is true since the relation $\rhd$ satisfies O.

  RI_3 We must show that if $\overline{[\psi]} \rhd \overline{[\phi]}$ and $\overline{[\chi]} \rhd \overline{[\psi]}$, then $\overline{[\chi]} \rhd \overline{[\phi]}$. This is true since the relation $\rhd$ is transitive.
- **Completeness :**
  We suppose that for every order relation $\rhd$ on 2^Ω satisfies O and Q, if $\forall i = 1 \cdots n, \overline{[\psi_i]} \rhd \overline{[\phi_i]}$ so $\overline{[\psi]} \rhd \overline{[\phi]}$. We must show that $(\mathcal{K}, >) \vdash_{\mathcal{S}} \phi > \psi$.
  If $\phi > \psi$ is in $(\mathcal{K}, >)$, it is proved.
  Otherwise, consider the strict partial order $\rhd$ defined on 2^Ω as the smaller relation containing the pairs $\overline{[\psi_i]} \rhd \overline{[\phi_i]}$ and closed for the properties Q, O and T. According to the hypothesis we have $\overline{[\psi]} \rhd \overline{[\phi]}$. And by definition of $\rhd$, the pair $(\overline{[\psi]}, \overline{[\phi]})$ is obtained by successive applications of properties Q, O and T. Using conjugated properties, this amounts to obtaining $\phi \succ \psi$ by successive applications of the rules RI_1, RI_2 and RI_3.

□

It is rather obvious that the inference relation $\vdash_{\mathcal{S}}$ does not lose any statement $\phi > \psi$ on the way, contrary to the inference using $\succ_N$ in example 2.

Example 3 *Let $(\mathcal{K}, >) = \{x > \neg x, y > \neg y\}$ be a partially ordered base.*
By RI_7 we have $x \wedge y > \neg x \vee \neg y$, than with RI_2 we obtain $y > \neg x$. And similarly we obtain $x > \neg y$. The inference using $\succ_N$ yields none of these results.

5.3 Inference based on level cuts

Another idea for inference from $(\mathcal{K}, >)$ could be inspired by possibilistic logic [10], namely, to work with level cuts of the partially ordered base, using classical logic. The

1. This axiom could be replaced by $\phi \vee \neg\phi > \psi \wedge \neg\psi$, in the presence of the rule RI_2.
2. The property T is also satisfied since $\rhd$ is a partial order

idea is to conclude $\phi > \psi$ when ϕ is classically deductible from a consistent set of formulas $\{\gamma_i \in \mathcal{K}\}$ such that $\forall i, \gamma_i > \psi$. This principle presupposes the axioms of Negligibility (N) and Orderliness (O) are valid for the relation $>$. In the following this inference is denoted by $\vdash_c$.

Definition 5 *Let $(\mathcal{K}, >)$ a partially ordered base. Let $\psi \in \mathcal{K}$, we define :*
- $\mathcal{K}^>_\psi = \{\gamma : \gamma \in \mathcal{K}$ *and* $\gamma > \psi\}$.
- $(\mathcal{K}, >) \vdash_c \phi > \psi$ iff $\mathcal{K}^>_\psi$ *is consistent and* $\mathcal{K}^>_\psi \vdash \phi$.

Note that the above definition presupposes that the relation $>$ is transitive.

Example 2 (continued) $\mathcal{K}^{\geq}_{\neg x} = \{x,\ x \wedge y,\ \neg x \vee y\}$ and $\mathcal{K}^{\geq}_{\neg x} \vdash y$ so $(\mathcal{K}, >) \vdash_c y > \neg x$.
$\mathcal{K}^{\geq}_{x \wedge y} = \{\neg x \vee y\}$ and $\mathcal{K}^{\geq}_{x \wedge y} \vdash \neg x \vee y$ so $(\mathcal{K}, >) \vdash_c \neg x \vee y > x \wedge y$.
The comparison $\neg x \vee y > x \wedge y$ has been preserved.

Example 4 *Let $\mathcal{K} = \{\neg x \vee \neg y > x \wedge y, x \wedge y > \neg x, x > \neg x\}$ a partially ordered base.*
$\mathcal{K}^{\geq}_{\neg x} = \{x,\ x \wedge y,\ \neg x \vee \neg y\}$. *This set of formulas is not consistent so $\mathcal{K}^{\geq}_{\neg x} \nvdash_c y > \neg x$.*

In the following, we compare the inference based on level cuts with the inference in the system $\mathcal{S}$. Let us consider the semantical point of view.

The following proposition expresses that the syntactic inference $\vdash_c$ (Definition 5) is sound for the semantics defined in Definition 4.

Proposition 6 *Let $(\mathcal{K}, >)$ a partially ordered base.*
For any formula $\psi \in \mathcal{K}$,
if $(\mathcal{K}, >) \vdash_c \phi > \psi$ then $(\mathcal{K}, >) \vDash_{\mathcal{S}} \phi > \psi$. The converse is false.

Proof of Proposition 6: We suppose that $(\mathcal{K}, >) \vdash_c \phi \succ \psi$. Then, by definition we have $\mathcal{K}^>_\psi \vdash \phi$ and $\mathcal{K}^>_\psi$ is consistent.
Let $\mathcal{K}^>_\psi = \{\gamma_1, \cdots, \gamma_p\}$. $\forall i, \gamma_i > \psi$. By definition of $\vDash_{\mathcal{S}}$, we assume that $\forall i, \overline{[\psi]} \rhd \overline{[\gamma_i]}$ and the relation $\rhd$ satisfies Q and O. So, it also satisfies the property N. So we have $\overline{[\psi]} \rhd \overline{[\gamma_1]} \cup \overline{[\gamma_2]} \cdots \overline{[\gamma_p]}$, or equivalently $\overline{[\psi]} \rhd \overline{[\gamma_1 \wedge \gamma_2 \cdots \gamma_p]}$.
As $\{\gamma_1, \cdots, \gamma_p\} \vdash \phi$, we have $[\gamma_1 \wedge \gamma_2 \cdots \gamma_p] \subseteq [\phi]$ so $\overline{[\phi]} \subseteq \overline{[\gamma_1 \wedge \gamma_2 \cdots \gamma_p]}$.
As the relation $\rhd$ satisfies O, we conclude that $\overline{[\psi]} \rhd \overline{[\phi]}$. □

Here is a counter-example for the converse :

Example 3 (continued)
$\overline{[\neg y]} \rhd \overline{[x]}$ but not $\mathcal{K}^{\geq}_{\neg y} \vdash x$ because $\mathcal{K}^{\geq}_{\neg y} = \{y\}$.
Indeed, by hypothesis we have $\overline{[\neg x]} \rhd \overline{[x]}$ and $\overline{[\neg y]} \rhd \overline{[y]}$.
The relation $\rhd$ satisfies O and Q then also CCC.
Due to CCC, we deduce that $[x \wedge y] \rhd \overline{[x \wedge y]} = [\neg x \vee \neg y]$.
Then due to O, we obtain $[y] \rhd [\neg x]$ be even $\overline{[\neg y]} \rhd \overline{[x]}$.
However, we do not have $\mathcal{K}^{\geq}_{\neg y} \vdash_c x$.

Due to Proposition 5 and Proposition 6, we conclude that $(\mathcal{K}, >) \vdash_c \phi > \psi$ strictly implies $(\mathcal{K}, >) \vdash_{\mathcal{S}} \phi > \psi$. Note that, contrary to the inference based on $\succ_N$, if $\phi > \psi$ is present in $(\mathcal{K}, >)$, we have that $\mathcal{K}^>_\psi \vdash \phi$ so that we do not lose information using the inference by cuts. It is moreover easy to implement. However its limitations stem from the fact that only statements $\phi > \psi$ where $\psi \in \mathcal{K}$ can be deduced.

6 Comparison between the different approaches

Our long range purpose is to compare the deductive closure obtained with the inference system $\mathcal{S}$ with inference in symbolic possibilistic bases. The first step is to investigate possible ways of encoding a possibilistic base with a partially ordered base and conversely.

6.1 Encoding a partially ordered base as a symbolic possibilistic base

A partially ordered base $(\mathcal{K}, >)$ does not involve absolute certainty weights. Encoding this base in the form a symbolic possibilistic base requires to attach symbolic weights to formulas of $\mathcal{K}$ and to write constraints on these weights. It may be considered as an enrichment of the original base as we shall see.

Formally, let $\eta : \mathcal{K} \longrightarrow L_{\mathcal{K}}$ a function that associates to each formula ϕ of $\mathcal{K}$ the symbol $\eta(\phi)$ of a weight. Then we build a set of constraints C such that $a > b$ if and only if $a = \eta(\phi), b = \eta(\psi)$ and $\phi > \psi$.

Definition 6 *Let $(\mathcal{K}, >)$ a partially ordered base. $(\mathcal{K}, >)$ is encoded by*
- $\Sigma_{\mathcal{K}} = \{(\phi, \eta(\phi)), \phi \in \mathcal{K}\}$
- $C = \{a > b / (\phi, a), (\psi, b) \in \Sigma_{\mathcal{K}}$ *and* $\phi > \psi \in \mathcal{K}\}$.

Example 5 *Let $(\mathcal{K}, >) = \{\neg x > \neg x \vee y, \neg y > \neg x \vee y\}$. $(\mathcal{K}, >)$ is encoded with $\Sigma_{\mathcal{K}} = \{(\neg x \vee y, a),\ (\neg x, b),\ (\neg y, c)\}$ and $C = \{c > a, b > a\}$.*

However, this kind of encoding may add some unwanted information, as shown by the following example.

Example 6 *Let $(\mathcal{K}, >) = \{\neg x \vee y > x \wedge y\}$. Let $a = \eta(\neg x \vee y)$ and $b = \eta(x \wedge y)$. We obtain $\Sigma_{\mathcal{K}} = \{(\neg x \vee y, a), (x \wedge y, b)\}$ and $C = \{a > b\}$. In possibilistic logic, believing each of two formulas to the same degree is equivalent to believing their conjunction to that degree. So, we can replace $(x \wedge y, b)$ by $(x, b), (y, b)$. Thus, we obtain $\Sigma = \{(\neg x \vee y, a), (x, b), (y, b)\}$ which is semantically equivalent to $\Sigma_{\mathcal{K}}$ (in the sense that $N_{\Sigma_{\mathcal{K}}} = N_\Sigma$, with symbolic weights).*
However, from $\neg x \vee y > x \wedge y$, we can deduce neither $\neg x \vee y > x$ nor $\neg x \vee y > y$ using inference system $\mathcal{S}$.

This remark suggests that inferring in symbolic possibilistic logic may be stronger than inference in system $\mathcal{S}$, since in possibilistic logic, $\{(\phi, a), (\psi \wedge \xi, b)\}$ is equivalent to $\{(\phi, a), (\psi, b), (\xi, b)\}$. However, under system $\mathcal{S}$, $\phi > \psi \wedge \xi$ is only implied by the conjunction $(\phi > \psi) \wedge (\phi > \xi)$.

6.2 Encoding a symbolic possibilistic base as a partially ordered base

Conversely, we propose one possible encoding of a symbolic possibilistic base (Σ, C) as a set of statements of the form $\phi > \psi$.
A possibilistic formula (ϕ, a) is interpreted as $N(\phi) \geq a$. So, given (ϕ, a) and (ψ, b) in Σ, with $a > b \in C$, a natural idea is to state that $\phi > \psi$. However, it may occur that in the deductive closure, $N_\Sigma(\psi) = b' > b$, as discussed in Section 2.1. So, it must be ensured that the formulas of Σ are assigned their maximum weight.
Let Σ^+ be the coherent base associated to (Σ, C) built as follows :

Definition 7 *Let (Σ, C) be a symbolic possibilistic base and $\Sigma^* = \{\phi : \exists a > 0, (\phi, a) \in \Sigma\}$ the corresponding set of formulas. The coherent base associated to (Σ, C) is*

$$\Sigma^+ = \{(\phi, N_\Sigma(\phi)) : \phi \in \Sigma^*\}.$$

Now, we build partially ordered formulas by comparing weights of formulas in Σ^+, using C.

Definition 8 *A symbolic possibilistic base (Σ, C) is encoded by :*
$(\mathcal{K}, >)_\Sigma = \{\phi > \psi : (\phi, w) \in \Sigma^+, (\psi, w') \in \Sigma^+$ *and* $C \vDash w > w'\} \cup \{\phi > \bot : (\phi, w) \in \Sigma^+$ *and* $C \vDash w > Inc(\Sigma)\}$.

Note that partially ordered bases $(\mathcal{K}, >)$ may not always contain statements of the form $\phi > \bot$, while partially ordered bases of the form $(\mathcal{K}, >)_\Sigma$ always will.

Example 7 *Let $\Sigma = \{(p, a), (\neg q, c), (\neg p, d), (q, e), (\neg p \vee q, b)\}$ and $C = \{a > b, b > d, d > e, a > c, c > e\}$.*

- $Inc(\Sigma) = N_\Sigma(\bot) = \max(d, \min(b, c))$
- $N_\Sigma(q) = \max(e, \min(a, b)) = b$.
  So $(q, b) \in \Sigma^+$
- $N_\Sigma(\neg p) = \max(d, \min(b, c))$.
  So $(\neg p, \max(d, \min(b, c))) \in \Sigma^+$.

Finally : $\Sigma^+ =$
$\{(\neg p, \max(d, \min(b, c))), (p, a), (\neg q, c), (\neg p \vee q, b), (q, b)\}$.
with $C \vDash a > Inc(\Sigma)$.
So $(\mathcal{K}, >)_\Sigma = \{(p > \bot), (p > \neg p), (p > \neg p \vee q), (p > q), (p > \neg q)\}$.

Note that we may have distinct, semantically equivalent Σ_1 and Σ_2 such that $(\mathcal{K}, >)_{\Sigma_1}$ and $(\mathcal{K}, >)_{\Sigma_2}$ are not semantically equivalent, due to the point made in Example 6. Conversely, $(\mathcal{K}, >)_{\Sigma_\mathcal{K}}$ is different from $(\mathcal{K}, >)$ since the latter may not contain $\phi > \bot$.

6.3 Comparison between possibilistic inference and syntactic methods

Starting from a symbolic possibilistic base (Σ, C), our purpose is to compare the possibilistic inference from (Σ, C) (Definition 1) with the syntactic inference from the partially ordered base $(\mathcal{K}, >)$ encoding (Σ, C). We first consider syntactic inference based on level cuts, then we consider syntactic inference with system $\mathcal{S}$.

Proposition 7 *Let (Σ, C) be a symbolic possibilistic base and $(\mathcal{K}, >)_\Sigma$ its encoding (according to Definition 8). Let $\psi \in \mathcal{K}$ be such that $\psi \neq \bot$, we have :*

$$(\Sigma, C) \vdash \phi > \psi \text{ iff } \mathcal{K}^>_\psi \vdash \phi.$$

Proof of Proposition 7:
We assume that $(\Sigma, C) \vdash \phi > \psi$, this means that :

- $N_\Sigma(\phi) = w$
- $N_\Sigma(\psi) = w'$
- $C \vDash w > w'$.

By definition of w', we have $w' \geq Inc(\Sigma)$. $\mathcal{K}^>_\psi = \{\phi \in \mathcal{K} : \phi > \psi\}$ and $\phi > \psi$ means that $(\phi, w") \in \Sigma^+$ and $C \vDash w" > w'$.
As $w' \geq Inc(\Sigma)$, $\mathcal{K}^>_\psi$ is a consistent set of formulas.
On the other hand, $\mathcal{K}^>_\psi$ contains Σ_w and $\Sigma_w \vdash \phi$. Thus, $\mathcal{K}^>_\psi \vdash \phi$.

Conversely, suppose $\mathcal{K}^>_\psi \vdash \phi$ and $\mathcal{K}^>_\psi$ is a consistent base.
Suppose $\mathcal{K}^>_\psi = \{\phi_i \in \mathcal{K} : \phi_i > \psi\} \vdash \phi$.
We know that if $\forall i, (\phi_i, w_i), (\psi, w') \in \Sigma^+$ and $C \vDash w_i > w'$, then $C \vDash \min_i w_i > w'$.
We have $N_\Sigma(\psi) = w'$. Hence $N_\Sigma(\phi) \geq \min_i w_i > w'$. Thus, $(\Sigma, C) \vdash \phi > \psi$.

□

There is no similar result even starting from any kind of partially ordered base of clauses (avoiding $\phi > \psi \wedge \xi$ cases). Indeed, in partially ordered bases, initial comparative statements $\phi > \psi$ are understood as hard constraints whereas in possibilistic logic constraints are of the form $N(\phi) \geq a$ and do not enforce such ordering between clauses.
Let $(\mathcal{K}, >)$ be a partially ordered set of clauses and $(\Sigma_\mathcal{K}, C)$ its encoding (according to Definition 6).
The following example illustrates that we do not have $\mathcal{K}^>_\psi \vdash \phi \Rightarrow (\Sigma_\mathcal{K}, C) \vdash \phi > \psi$.

Example 5 (continued)
$(\mathcal{K}, >) = \{\neg x > \neg x \vee y, \neg y > \neg x \vee y\}$.
$(\mathcal{K}, >)$ is encoded by :
$\Sigma_\mathcal{K} = \{(\neg x \vee y, a),\ (\neg x, b),\ (\neg y, c)\}$
and $C = \{c > a, b > a\}$. $\mathcal{K}^>_{\neg x \vee y} = \{\neg x, \neg y\} \vdash \neg x$.
Take $\phi = \neg x$ and $\psi = \neg x \vee y$.
$N_{\Sigma_\mathcal{K}}(\psi) = \max(a, b) = b$
$N_{\Sigma_\mathcal{K}}(\phi) = \max(b, \min(a, \gamma)) = b$.
Thus, we do not have $(\Sigma_\mathcal{K}, C) \vdash \phi > \psi$.

Now we consider syntactic inference with system $\mathcal{S}$. We show that it is more demanding than possibilistic inference.

Proposition 8 *Let (Σ, C) be a symbolic possibilistic base and $(\mathcal{K}, >)_\Sigma$ its encoding (according to Definition 8).*

$$(\mathcal{K}, >)_\Sigma \vdash_\mathcal{S} \phi > \psi \Rightarrow (\Sigma, C) \vdash \phi > \psi$$

Proof of Proposition 8:
We assume that $(\mathcal{K}, >)_\Sigma \vdash_\mathcal{S} \phi > \psi$.
The proof is by induction on the number of steps using the inference rules of the system $\mathcal{S}$, RI_1, RI_2, RI_3 and RI_4 :

Case when $\phi > \psi \in (\mathcal{K}, >)_\Sigma$: it means that $(\phi, a) \in \Sigma^+, (\psi, b) \in \Sigma^+$ and $C \vDash a > b$.
Or equivalently $(\Sigma, C) \vdash (\phi, a), (\Sigma, C) \vdash (\psi, b)$ and $C \vDash a > b$ (which can be written $N(\phi) > N(\psi)$). That is exactly the definition of $(\Sigma, C) \vdash \phi > \psi$.

If RI_1 is applied : We have $\phi = p \wedge q$ and $\psi = r$
With $(\mathcal{K}, >)_\Sigma \vdash_\mathcal{S} q > p \wedge r$ and $(\mathcal{K}, >)_\Sigma \vdash_\mathcal{S} p \wedge p > q \wedge r$.
By induction hypothesis :

$$N(q) > N(p \wedge r) = \min(N(p), N(r))$$

$$N(p) > N(q \wedge r) = \min(N(q), N(r))$$

If $N(r) > N(p)$ then $N(q) > N(p)$. Thus, $\min(N(q), N(r)) > N(p)$ (impossible).
Thus $N(p) \geq N(r)$.
So $N(q) > N(r)$ and $N(p) > N(r)$.
Hence, $N(p \wedge q) > N(r)$.
Thus $N(\phi) > N(\psi)$.

If RI_2 is applied : We have $(\mathcal{K}, >)_\Sigma \vdash_\mathcal{S} \phi' > \psi'$ and $\phi' \vDash \phi$ and $\psi \vDash \psi'$.
By hypothesis $N(\phi') > N(\psi')$. We have also $N(\phi) \geq N(\phi')$ and $N(\psi') \geq N(\psi)$. Thus, $N(\phi) > N(\psi)$.

If RI_3 or RI_4 is applied : the strict comparison using $\succ_N$ is transitive and irreflexive.

□

The following example shows that the converse is false.

Example 8 *Let $\Sigma = \{(p, a), (q, b), (r, c)\}$ and $C = \{a > b, a > c\}$. We have $(\Sigma, C) \vdash (q \vee r, \max(c, b))$. So $(\Sigma, C) \vdash (p > q \vee r)$.*
Encoding the possibilistic base produces the partially ordered base $(\mathcal{K}, >)_\Sigma = \{p > \bot, q > \bot, r > \bot, p > q, p > r\}$ from which we can not deduce $p > q \vee r$, using the inference system $\mathcal{S}$.

7 Related works

Many approaches dealing with inference from a partially ordered base have been proposed in the literature. A first approach was studied by Benferhat and Prade [2] which is the possibilistic approach restricted to plausible reasoning where only formulas above the inconsistency degree are deduced. But this approach seems to address a different problem ; especially, it allows to deduce only single formulas not a couple of the form $\phi > \psi$.

The second approach has been proposed by Yahi and al [16]. A partially ordered base $(\mathcal{K}, >)$ is viewed as a set of possible stratifications of $\mathcal{K}$ (totally ordered bases). So $\psi \geq \phi$ of $(\mathcal{K}, >)$ means that ψ is more certain than ϕ (in the sense of possibilistic logic) in all the stratified bases compatible with $(\mathcal{K}, >)$. Results in [4] indicate the strong link between this view and the weak optimistic relation $\succ_{wos}$. However, it is not an approach that is easy to implement due to the necessity of enumerating total orders compatible with a partial one. But the inferential power of this approach needs more scrutiny.

A third approach is closer to our semantic. Lewis [14] and Fariñas et al. [5] and Halpern [12] consider atomic expressions of the form $\phi \succ \psi$ as the basic syntactic entities of the language encoding preference.Specific axioms and inference rules which come from properties of the order relation are used. But these approaches rely on a richer language for handling atomic propositions of this language (using conjunctions, disjunctions and negations). However, one may wish to restrict the inference machinery to useful consequences of the form $\phi \succ \psi$ and $\phi \succeq \psi$ (only negation of atoms and conjunction of formulas), and so refrain from using disjunctions. Otherwise the deductive closure contains hardly interpretable statements.

In the fourth approach [1], the partial order on $\mathcal{K}$ is just used to select preferred consistent subsets of formulas, and the deductive closure is a classical set of accepted beliefs. So, as pointed out in Benferhat and Yahi [3], the deductive closure of a partially ordered base $(\mathcal{K}, >)$ is just a deductively closed set (in the classical sense), obtained from preferred subbases. Then the inference $(\mathcal{K}, >) \vdash \phi$ expressed that ϕ is consequence of all the preferred subsets of formulas.

8 Conclusion

This paper is another step in the study of inference from a partially ordered propositional base. After reminding possibilistic logics, and its version with constrained symbolic weights, our paper first explains how to construct a partial ordering on models from a partial order on formulas and back, thus defining a semantic notion of deductive closure in the spirit of possibilistic logic. However, it turns out that this approach is not faithful and we may loose information in the partially deductive closure. To overcome this drawback we have proposed to move from a partial order between interpretations to a partial order between subsets thereof. It presupposes the choice of properties that the partial order between formulas is supposed to have. Here we interpret the partial order in terms of relative certainty, in a qualitative setting faithful to possibility theory. We have proposed an inference system inspired from previous conditional logics proposed by Lewis [14], Fariñas et al. [5] and Halpern [12], however simplified to produce only a partial order over the whole language. We have started a comparison between this approach and inference in possibilistic logic with symbolic weights.

A similar analysis could be carried out for preadditive partial orders (that are self-conjugate like probability relations) [4].

This work has potential applications for the revision and the fusion of beliefs, as well as preference modeling [11].

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